Statistical Inference
Test Set 5

1. In the following hypotheses testing problems identify the given hypotheses as simple or composite.
   (i) $X \sim Exp(\lambda), H_0 : \lambda \leq 1$
   (ii) $X \sim Bin(n, p), n$ is known, $H_0 : p = 1/3.$
   (iii) $X \sim Gamma(r, \lambda), r$ is known , $H_0 : \lambda > 2$.
   (iv) $X_1, \cdots, X_n \sim N(\mu, \sigma^2), H_0 : \mu = 0, \sigma^2 = 2.$

2. Let $X \sim P(\lambda)$. For testing $H_0 : \lambda = 1$ vs. $H_0 : \lambda = 4$ consider the test $\phi(x) = 1,$ if $x > 2;$ and 0, if $x \leq 2.$ Find the probabilities of type I and type II errors.

3. Let X have double exponential density $f_X(x) = \frac{1}{2\sigma} e^{-|x|/\sigma}, x \in \mathbb{R}, \sigma > 0.$ For the testing $H_0 : \sigma = 1$ vs. $H_0 : \sigma > 1$ consider the test function $\phi(x) = 1,$ if $|x| > 1;$ and 0, if $|x| \leq 1.$ Find the size and the power of the test. Show that power is always more than the size of the test.

4. Let X have Cauchy density $f_\sigma(x) = \frac{\sigma}{\pi(\sigma^2 + x^2)}, x \in \mathbb{R}, \sigma > 0.$ Find the most powerful test of size α for the testing $H_0 : \sigma = 1$ vs. $H_0 : \sigma = 2$.

5. Let X have density $f_\theta(x) = \frac{2}{\theta^2}(\theta - x), \quad 0 < x < \theta$. Find the most powerful test of size α for the testing $H_0 : \theta = 1$ vs. $H_0 : \theta = 2$.

6. Let $X_1, \cdots, X_n$ be a random sample from a population with density $f_\theta(x) = \frac{1}{\Gamma\theta} x^{\theta-1} e^{-x}, \; x > 0, \theta > 0.$ Show that the family has monotone likelihood ratio in $\prod X_j$. Hence derive UMP test of size α for testing $H_0 : \theta \leq 3$ vs. $H_0 : \theta > 3$.

7. Let $X_1, \cdots, X_n$ be a random sample from a population with beta density $f_\theta(x) = \frac{\Gamma\theta + 4}{2\Gamma\theta + 1} x^\theta (1 - x)^2, \; 0 < x < 1, \theta > 0.$ Show that the family has monotone likelihood ratio in $\prod X_j$. Hence derive UMP test of size α for testing $H_0 : \theta \geq 2$ vs. $H_0 : \theta < 2$.

8. Based on a random sample of size n from $Exp(\lambda)$ population, derive UMP unbiased test of size α for testing $H_0 : \lambda = 1$ vs. $H_0 : \lambda \neq 1$.

9. Based on a random sample of size n from double exponential population with density $f_X(x) = \frac{1}{2\sigma} e^{-|x|/\sigma}, x \in \mathbb{R}, \sigma > 0$ derive UMP unbiased test of size α for testing $H_0 : \sigma = 1$ vs. $H_0 : \sigma \neq 1$.

10. For the set up in Q. 5, find the LRT for the testing $H_0 : \theta = 2$ vs. $H_0 : \theta \neq 2$.

11. Derive LRT test for the testing problem in Q. 9.

12. Let $X_1, X_2, ..., X_n$ be a random sample from an exponential population with the density $f(x) = e^{\mu - x}, \; x > \mu, \mu \in \mathbb{R}$. Find the LRT of size α for testing $H_0 : \mu \leq 1$ vs $H_1 : \mu > 1$.

13. Find out the group of transformations under which the following testing problems are invariant:
    (i) $X \sim e^{\theta - x}, \theta > x, H_0 : \theta \geq 3$ vs. $H_1 : \theta < 3$
    (ii) $X \sim Exp(1/\sigma), H_0 : \sigma \leq 1$ vs. $H_1 : \sigma > 1$

14. Let $X_1, X_2, ..., X_n$ be a random sample from an inverse Gaussian distribution with density $f_X(x) = \left(\frac{\lambda}{2\pi x^3} \right)^{1/2} \exp\left\{ -\frac{\lambda (x-\mu)^2}{2\mu^2 x} \right\}, x > 0$. Find the confidence intervals for the parameters.

15. Let $X_1, X_2, ..., X_n$ be a random sample from a Pareto population with density $f_X(x) = \frac{\beta \alpha^\beta}{x^{\beta+1}}, x > \alpha, \alpha > 0, \beta > 2.$ Find the confidence intervals for α, β.

Hints and Solutions

1. (i) composite (ii) simple (iii) composite (iv) simple
2. $\alpha = P_{\lambda=1}(X > 2) = 1 - \frac{5}{2}e^{-1}$, $\beta = P_{\lambda=4}(X \le 2) = 13e^{-4}$
3. $\alpha = P(|X| > 1) = e^{-1}$. Power = $P_{\sigma}(|X| > 1) = e^{-1/\sigma} > e^{-1}$, as $\sigma > 1$.
4. Using NP Lemma, the most powerful test is to reject H_0 when $R(x) = \frac{f_2(x)}{f_2(x)} > k$. Now $R(x) = \frac{2(1+x^2)}{(4+x^2)}$. It can be seen that $R'(x) = \frac{6x}{(4+x^2)^2}$. So $R(x)$ has a minimum $\frac{1}{2}$ at $x = 0$ and supremum 2 as $x \to \pm\infty$. The most powerful test can then be designed as below:
   (i) If we take $k \le \frac{1}{2}$, then the MP test will always reject H_0 and $\alpha = 1$.
   (ii) If we take $k \ge 2$, then the MP test will always accept H_0 and $\alpha = 0$.
   (iii) If we take $\frac{1}{2} < k < 2$, then the MP test will reject H_0 when $R(x) > k$. This is equivalent to $|x| > \sqrt{\frac{(4k-2)}{(2-k)}}$. Applying the size condition, we get $k = \frac{4}{4 + 3\cos(1-\alpha)}$.

5. Using NP Lemma, we find that the MP test will reject H_0 when $x < 1 - \sqrt{1-\alpha}$.
6. The solution will follow from the use of MLR property.
7. The solution will follow from the use of MLR property.
8. The test will be based on $\sum_{i=1}^{n} X_i$ which has a *Gamma* (n, λ) distribution.
9. The test will be based on $\sum_{i=1}^{n} |X_i|$ which has a *Gamma* $(n, 1/\sigma)$ distribution.
10. The test will reject H_0 when $|X - 1| > k$.
11. Use property in Q. 9.
12. The test will reject H_0 when $X_{(1)} > 1 - \ln(\alpha)^{1/n}$.
13. (i) translation group (ii) scale group
14. We can use the complete sufficient statistics and their distributions to formulate the confidence intervals.
15. Same method as in Q. 14.